# Announcements

Next plans: Spring break

- Monday/Tuesday April 6-7 section on divide and conquer, no quiz
- HW8 will be divide and conquer, due Friday April 10

Many office hours cancelled this week

Join by Web PollEv.com/evatardos772

How was the prelim?

A. Challenging
B. OK
C. OK, but too long, I ran out of time
D. I found it easy

# Convolution and its applications

multiply polynomials

$$f(x) = a_0 + x a_1 + x^2 a_2 + \dots a_n x^n$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \dots b_m x^m$$

$$h(x) = f(x) \cdot g(x) = c_0 + c_1 x + \dots + c_{n+m} x^{n+m}$$

$$\boxed{c_k = \sum_{\substack{i,j \\ i+j=k}} a_i b_j}$$

$$c_0 = a_0 b_0$$

$$c_{n+m} = a_n b_m$$

$$c_1 = a_0 b_1 + a_1 b_0$$

$$c_2 = a_0 b_2 + a_1 b_1 + a_2 b_2$$

convolution

input sequences $a_0 \dots a_n$

$b_0 \dots a_m$

compute $c_k = \sum_{i+j=k} a_i b_j \quad k = 0, \dots, n+m$

# More applications

two dice $\Pr(\text{sum}=7) = \sum_{\substack{i,j \\ i+j=7}} \Pr(\text{dice } 1 = i)\Pr(\text{dice } 2 = j)$

dice independent

assuming scores on prelim questions indep

$\Pr(\text{quest } i = q_i) = p_i \qquad i = 3, 4$

$\Pr(\text{total score q } 3\,4 = s) = \sum_{p_3 + p_4 = s} \Pr(\text{prob } 3 = p_1)\Pr(\text{prob } 4 = p_2)$

# Polynomial multiplication via divide and conquer (Karatsuba)

$f(x) = a_0 + a_1 x + \dots \quad a_n x^n$

$g(x) = b_0 + b_1 x + \quad b_m x^m$

$h(x) = f(x) g(x) = c_0 + c_1 x + \dots$

$\underbrace{a_0 \ a_1 \dots a_{n/2}} \quad \underbrace{a_{n/2+1} \dots a_n}$

$f(x) = f_1(x) + x^{n/2+1} f_2(x)$

degree $n/2$ degree $\leq n/2$

$g(x) = g_1(x) + x^{\frac{n}{2}+1} g_2(x)$

assume $m = n$

Can do like integer multiplication

$h(x) = \underline{f_1(x) g_1(x)} + x^{n/2+1}(f_1(x) g_2(x) + f_2(x) g_1(x))$

$x^{n+2} \underline{f_2(x) g_2(x)}$

get middle with

$[f_1(x) + f_2(x)][g_1(x) + g_2(x)] - f_1(x) g_1(x)$

$- f_2(x) g_2(x)$

time $T(n) = 3T(n/2) + O(n) \implies T(n) = O(n^{\log_2 3})$

for addition & subtraction

# Idea of faster polynomial multiplication via evaluations

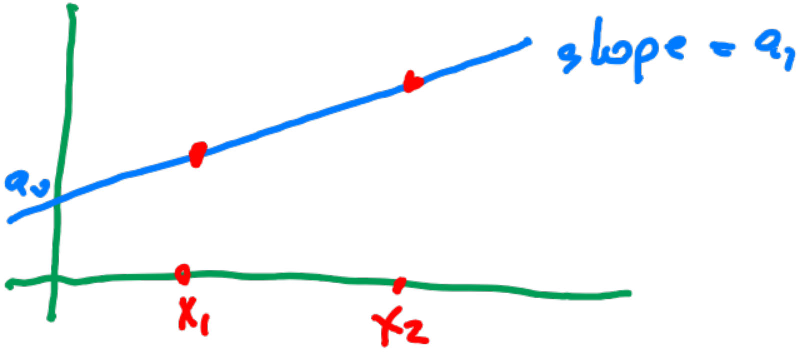
way to give polynomials of degree $n$

1. $f(x) = a_0 + a_1 x + \dots + a_n x^n$   coefficients

2. given $x_0 \dots x_n$   $f(x_i) = f_i$ for all $i$

need values at $n+1$ points to determine a polynomial

e.g $n=1$   $a_0 + a_1 x$




Idea: agree / points $x_0 \dots x_n$

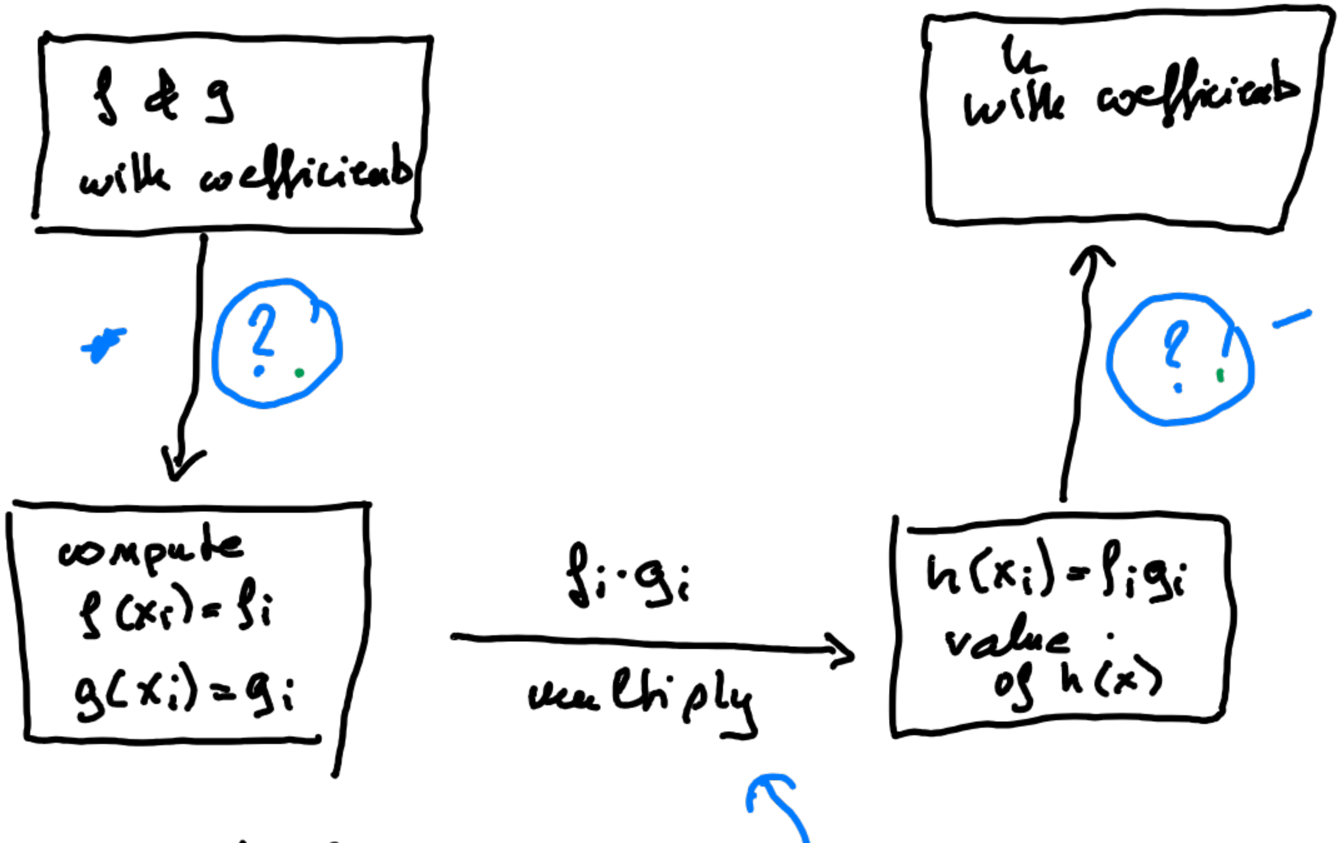
$f \longrightarrow f_i = f(x_i) \quad \forall i$   degree $n$

$g \longrightarrow g_i = g(x_i) \quad \forall i$   degree $m$

$h = f \cdot g$   $h(x_i) = f(x_i) g(x_i) = f_i g_i$

# points needed $n+m+1$ points

Plan given $f(x)$ degree $n$ & $g(x)$ degree $m$




$i = 0, \dots, n+m$

running time
$O(n+m)$ multiplications

next to do:

how to do ? parts
& what choice of $x_i$ makes them easy

given $x_0 \dots x_k$
compute $f(x_i)$ all $i$ $\quad f(x) = a_0 + \dots a_n x^k$ } $O(nk)$ with naive version

naive version:
given one $x_i$ $\quad a_0 + a_1 x_i + a_2 x_i^2 + a_3 x_i^3$ $\quad O(n)$ time

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$$f(x) = \underbrace{a_0} + a_1 x + \underbrace{a_2 x^2} + a_3 x^3 + \dots$$

$$f_2(x^2) = a_0 + a_2 x^2 + a_4 x^4 + \dots$$

$$f_2(x) = \sum_{i=0} a_{2i} x^i$$

$$\Rightarrow f_2(x^2) = \sum_{i=0} a_{2i} (x^2)^i$$

$$= \sum_{i=0} a_{2i} x^{2i}$$

$$a_1 x + a_3 x^3 + a_5 x^5 + \dots$$

$$x f_1(x^2) = x(\underbrace{a_1 + a_3 x^2 + a_5 x^4 + \dots})$$

$$f_1(x) = \sum_{i=0} a_{2i+1} x^i$$

$$x f_1(x^2) = \sum_{i=0} a_{2i+1} (x^2)^i x$$

$$= \sum_{i=0} a_{2i+1} x^{2i+1}$$

Result $f(x) = f_2(x^2) + x f_1(x^2)$

$f$ degree $n$ $\Rightarrow$ $f_1$ & $f_2$ both degree $n/2$

2 recursive problems + $O(n)$ time

Idea: find $x_0 \dots x_k$ so that $x_i^2$ has only ~ 1/2 as many

see next class